Chapter 2

Electrostatics

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2.1 Coulomb's Law

This course is a study of that substance in nature that has come to be known as *charge*. This character, or property possessed by certain atomic particles, manifests itself in the macroscopic world via a force. It is observed that there are two distinct types of charge which are traditionally referred to as positive (+) and negative (-). Most substances hold a roughly equal number of each charge type so that they appear to be *neutral* to the observer. When an isolated solid, liquid or gas holds an excess of one charge type over the other, we say that the substance is *charged*. Typically, one always refers to the excess charge within or on a material medium as the charge held by the material and henceforth in this text it will be assumed that a non-zero charge refers to an excess charge of one type or the other.

The SI unit for charge is the Coulomb (C). For our purposes in this text, the fundamental source of all negative charge is the electron which holds the charge -1.6×10^{-19} C. Likewise the proton is the source of all positive charge with a charge per particle of $+1.6 \times 10^{-19}$ C. With respect to charge, most material mediums can be classified as being one of two types, *conductors* or *insulators*. Roughly speaking, charge can pass freely through a conductor whereas charge cannot flow through an insulator.

In the late 18$^{\text{th}}$ century the French scientist Charles Coulomb (1736-1806), using a delicate apparatus, was able to quantify the nature of the force between charged particles. These results indicated that the magnitude of the force, F, between two point charges is directly proportional to the absolute values of the product of the charges,

$$F \propto |q_1 q_2| \ , \tag{2.1}$$

where q_1 is the total excess charge on point charge 1 and q_2 is the total excess charge held on point charge 2.

Additionally, Coulomb found that this force magnitude was inversely proportional to the square of the distance, r, between the particles' centers. That is,

$$F \propto \frac{1}{r^2} \ . \tag{2.2}$$

Using these two relations together, along with a constant of proportionality, k, leads to *Coulomb's law*:

$$F = k\frac{|q_1 q_2|}{r^2} \ , \tag{2.3}$$

Coulomb also found that the force was directed along a line between the centers of the charged particles and

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that like charges repel and unlike charges attract. This gives us a sense of direction so that we can write an expression for the force on one charge due to the other as a vector quantity.

$$\vec{F} = k\frac{|q_1 q_2|}{r^2}\hat{r} \ . \tag{2.4}$$

Here $\hat{r}$ is a unit vector in the required direction. Forces that are described by Eq. (2.4) are referred to as *Coulomb forces*.

Since forces come in equal and opposite pairs, we always take the perspective of finding the force on one charge due to another. As an example, consider two charges fixed in space as shown in Figure 2.1. We refer to such charges as *static* charges. The force between these two charges of opposite sign would be attractive. We can write an expression for the force on q_1 due to q_2, $\vec{F}_{12}$, by using the unit vector shown in Figure (2.1), in Eq. (2.4).

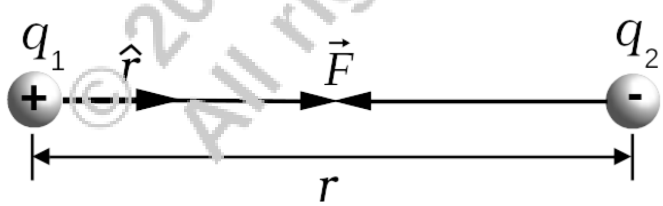

Figure 2.1: Two static charges separated by a distance r experience a Coulomb force $\vec{F}$.

Assuming the direction and notation for Cartesian coordinates we can put all of this together as

$$\vec{F}_{12} = k\frac{|q_1 q_2|}{r^2}\hat{x} \ . \tag{2.5}$$

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In SI units, k takes the form:

$$k = \frac{1}{4\pi\epsilon_o} \ , \tag{2.6}$$

where ϵ_o is called the *permittivity of free space*. This constant has the value $\epsilon_o = 8.85 \times 10^{-12}$ $\mathrm{C}^2/(\mathrm{N\ m}^2)$. Its presence within our formula for Coulomb's law indicates that all of our charges reside in free space, (vacuum) or air as air very nearly has the same properties as free space when it comes to electric fields. All of our charges in this chapter can be assumed to exist in free space. We will have more to say about electric fields in material mediums in a later chapter. It should be pointed out that Eq. (2.5) is valid only when the particles are point charges or when the distance between the charged objects is much greater than their size.

It should be noted that in all cases in this text where a Coulomb force is computed, any force due to gravity is ignored. It can be shown, for any charged object we will consider, that the Coulomb force is much greater in magnitude than any gravity force involved and thus gravity forces are ignored.

Now the question arises of how to deal with the force on one fixed charge due to multiple static charges in the surrounding area? The net force acting on the charge of interest can easily be found if we assume that the *principle of superposition* holds. Here it is assumed that all of the forces between pairs of particles act independently of one another. Then, the net force, $\vec{F}$, acting on the charge of interest q_1 is given by the sum of the forces of all N charges acting acting independently upon q_1. That is,

$$\vec{F} = \vec{F}_{12} + \vec{F}_{13} + \vec{F}_{14} + \cdots + \vec{F}_{1N} \ . \tag{2.7}$$

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We can use position vectors, and the superposition principle, to rewrite Eq. (2.5) in a more general way which gives the force on charge q_1 due to N static point charges. The required position vectors for describing the force between q_1 and q_2 are shown in Figure 2.2. Though this is a two dimensional figure the result will be valid for three dimensions as well.

Let the position vector to q_1 be $\vec{r}_1$ and the position vector to q_1 be $\vec{r}_2$. Then, a vector that points from q_2 to q_1 is $\vec{r}_1 - \vec{r}_2$. Therefore, the straight line distance between q_2 and q_1 must be

$$|\vec{r}_1 - \vec{r}_2| \ . \tag{2.8}$$

A position vector, $\hat{r}_{21}$, that points from q_2 to q_1 is

$$\hat{r}_{21} = \frac{\vec{r}_1 - \vec{r}_2}{|\vec{r}_1 - \vec{r}_2|} \ . \tag{2.9}$$

Using such vectors for all N charges in the static assembly, along with the straight line distance distance between them, Eq. (2.5) can be adapted to give the force on q_1 due to an assembly of N static charges.

$$\vec{F} = kq_1 \sum_{n=2}^{N} \frac{q_n \left(\vec{r}_1 - \vec{r}_n\right)}{|\vec{r}_1 - \vec{r}_n|^3} \ . \tag{2.10}$$

Example 2.1

A system of four static charges, all of identical charge q, are placed in a two dimensional system as: One at origin, one at (0,1), one at $(\sqrt{2}/2, \sqrt{2}/2)$ and the final one at (1,0). Find the force on the charge at origin due to the other three. All units are SI.

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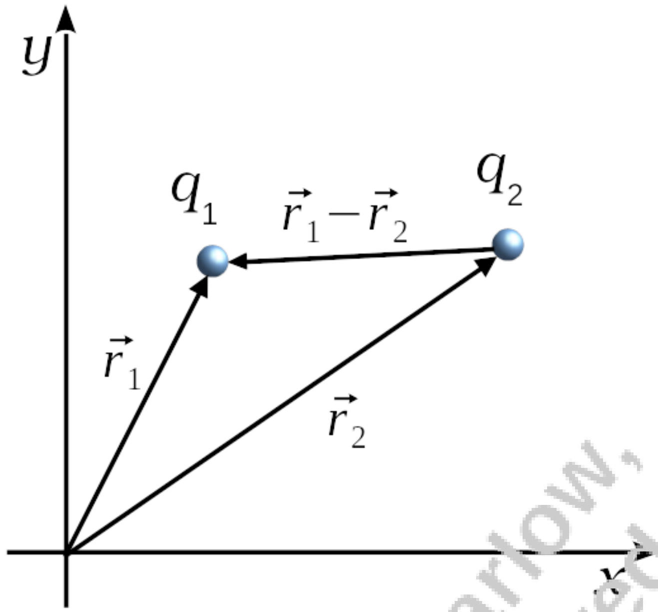

Figure 2.2: Two static charges with position vectors denoting locations.

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We apply Eq. (2.10). Here $\vec{r}_1 = 0$. Assemble the other required vectors:

$\vec{r}_2 = \hat{y}$.
$\vec{r}_3 = \sqrt{2}/2\ \hat{x} + \sqrt{2}/2\ \hat{y}$
$\vec{r}_4 = \hat{x}$.

Now, we use these vectors and their magnitudes in Eq. (2.10) for the case where $N = 4$:

$\vec{F} = kq^2 \left(\frac{-\hat{y}}{1^3} - \frac{\sqrt{2}/2\ \hat{x} + \sqrt{2}/2\ \hat{y}}{1^3} - \frac{\hat{x}}{1^3} \right)$.

Combining like terms leads to

$$\vec{F} = -kq^2 \left[(1 + \sqrt{2}/2)\ \hat{x} + (1 + \sqrt{2}/2)\ \hat{y} \right] \ .$$

PROBLEMS

All charges are static in the following problems.

2.1 A crude model for the helium atom has a proton of charge $+2q$ at the origin of a two dimensional system while one electron of charge $-q$ is located at $(0, d)$ and the other at $(a, 0)$. Assume all SI units. What is the force on the electron at $(0, d)$ due to the other two charged particles? What does this force become when $a \to \infty$?

2.2 Consider a point charge q placed at origin. There are an infinite number of equivalent point charges placed on the x axis to the right of this charge at positions $(n-1, 0)$ for $n = 2, 3, 4, ...\infty$. Show that the net force acting on the charge at the origin is

$$\vec{F} = -\frac{kq^2\pi^2}{6}\,\hat{x}\ .$$

You will need the identity:

$$\sum_{n=2}^{\infty} \frac{1}{(n-1)^2} = \frac{\pi^2}{6}\ .$$

2.3 In the hydrogen atom, the distance between the proton and the electron is about 0.5 Å. Compute the Coulomb force and the gravity force for this arrangement then the ratio of these forces. Comment on the result.

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2.2 The Electric Field

You may already have concluded that the procedures described in the preceding section would be tedious to apply if the number of static charges were large. This is obviously the case for most charged objects as they typically hold an enumerable number of charged atomic particles.

Much of this difficulty can be overcome by dealing with a charge density rather than a group of discrete point charges. Then, Eq. (2.10) could be replace by an integral equation.

Before we do this though it is useful to introduce the idea of an *electric field*. Often our static charge distributions will be in the form of regular arrangements that are stable. For example, a flat charged plate, a charged sphere or a straight line of charge. Since such arrangements occur so often it is useful to be able to quickly analyze how these distributions interact with other charged particles that might be in their vicinity. The electric field, which is a vector field, is a construct that enables such a procedure. The electric field is a permanent characteristic of the fixed charge distribution. With knowledge of this field, any free charge that enters into the field experiences a Coulomb force that can be easily computed with out resorting to expressions like Eq. (2.10).

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To obtain an expression for the electric field due to some fixed assembly of charges, a positive test charge, $+q$, is placed in the vicinity of the distribution. The test charge will experience a Coulomb force $\vec{F}$. The electric field vector at this point, $\vec{E}$, is then defined by

$$\vec{F} = q\vec{E} \ . \tag{2.11}$$

The interesting feature of Eq. (2.11) is that $\vec{E} = \vec{F}/q$ so that the electric field does not include the field due to the test charge. It is a permanent characteristic of the fixed charge distribution alone. It can be seen from Eq. (2.11) that the SI units for the electric field are N/C.

It would be a straight forward thing to modify Eq. (2.10) so that it can be used to compute the electric field vector at some point, P, due to a static charge distribution. However, as mentioned earlier, it is a lot easier to deal with a density of charge rather than individual point charges. Often, the density of charge, whether on a line, surface or in a volume is known or can be reasonable estimated. In this case, the sum in Eq. (2.10) is replaced with an integral. This charge density, can be uniform or vary with position.

To illustrate the process of computing the electric field due to a static charge density, consider the electric field differential $d\vec{E}$, at some point P, due to some infinitesimal charge, dq, within some region of volume charge density ρ. This situation is depicted in Figure 2.3. Using Eq. (2.11)

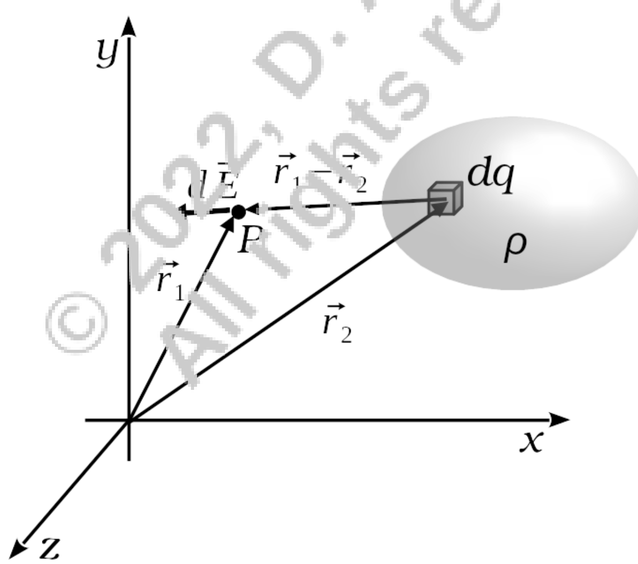

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Figure 2.3: Position vectors required for finding $d\vec{E}$ at point P due to a charge distribution ρ.

in Eq. (2.10) letting $q = q_1$, $q_2 = dq$ where $dq = \rho d\tau$, gives

$$d\vec{E} = k\frac{\rho d\tau\,(\vec{r}_1 - \vec{r}_2)}{|\vec{r}_1 - \vec{r}_2|^3}\ . \tag{2.12}$$

Integrating over the volume of the charge τ then yields the complete electric field vector at the point P.

$$\vec{E} = k\int\limits_{\tau} \frac{\rho d\tau\,(\vec{r}_1 - \vec{r}_2)}{|\vec{r}_1 - \vec{r}_2|^3}\ . \tag{2.13}$$

Of course the challenging aspect of applying Eq. (2.13) is to make sure that the vectors, ρ and the differential $d\tau$, are all correct for the coordinate system involved.

Often, Eq. (2.13) can be simplified by orienting the region of charge in a favorable location in the coordinate system. When such accommodations can be made, it becomes possible to quickly write the vector $\vec{r}$, directed from dq to the point of interest, P, without resorting to finding the difference of two vectors. Letting $|\vec{r}| = r$ and $\hat{r}$ be a unit vector in the direction of $\vec{r}$ then Eq. (2.13) simplifies to

$$\vec{E} = k\int\limits_{\tau} \frac{\rho d\tau}{r^2}\ \hat{r}\ . \tag{2.14}$$

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To illustrate this process, let us consider a thin rod of length l which holds an excess positive charge given by a uniform linear charge density λ. Therefore, the total charge on the rod will be λl. Suppose we want to determine an expression for the electric field a distance y above the middle of this rod. This situation is depicted in Figure 2.4.

We want the differential for $\vec{E}$, $d\vec{E}$, due to the charge differential dq. This is accomplished by setting up dq, and the required vectors correctly, in Eq. (2.12). But, in this situation it's easy, we need only use the differential form of Eq. (2.14) since it's straight forward to write $\hat{r}$ and r for this arrangement. Then, we simply add these differentials up across the rod to get the desired result. According to our figure, $dq = \lambda dx$. Putting it all together we get,

$$d\vec{E} = \frac{\lambda dx}{4\pi\epsilon_o(x^2+y^2)} \frac{(x\ \hat{x} + y\ \hat{y})}{\sqrt{x^2+y^2}} \ . \tag{2.15}$$

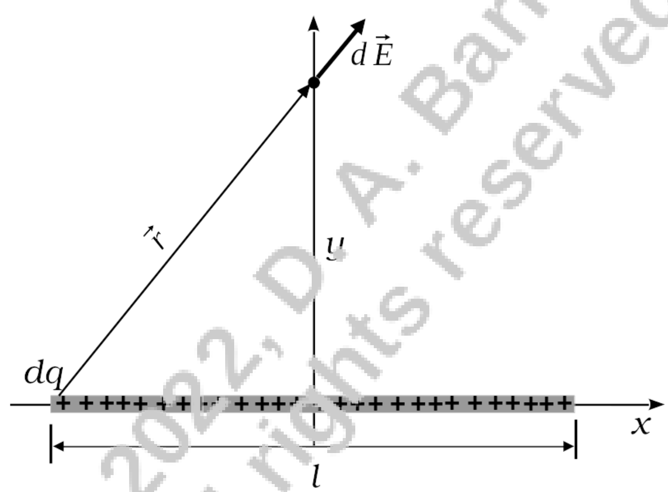

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Figure 2.4: Uniform line charge of length l with uniform linear charge density λ.

Before proceeding further let's discuss the central features of Eq. (2.15). The distance between dq and q is, for any position x along the rod $\sqrt{x^2+y^2}$ so that the square of this gives (x^2+y^2). The vector on the far right side of the expression gives a unit vector along the direction of the force between dq and a test charge placed at point P, valid for any position, x, along the rod. We can now integrate both sides of Eq. (2.15) along the length of the rod and get our answer.

However, it is useful to use symmetry here to simplify the integration. Due to symmetry, all of the horizontal components of $\vec{E}$ will cancel. That is, the horizontal components on the left side will cancel those on the right. With this in mind, the $\hat{x}$ component of the unit vector in Eq. (2.15) can be ignored. We can then integrate both sides of Eq. (2.15) as

$$\vec{E} = \frac{\lambda}{4\pi\epsilon_o} \int_{-l/2}^{l/2} \frac{dx}{(x^2 + y^2)^{3/2}} \; \hat{y} \; . \tag{2.16}$$

Notice how when $\sqrt{x^2 + y^2}$ and $(x^2 + y^2)$ are combined, we get $(x^2+y^2)^{3/2}$. After consulting the table of integrals in an Appendix, one finds that the above leads to

$$\vec{E} = \frac{\lambda}{4\pi\epsilon_o} \left. \frac{x}{y(x^2 + y^2)^{1/2}} \right|_{-l/2}^{l/2} \hat{y} = \frac{\lambda l}{4\pi\epsilon_o z \left[\left(\frac{l}{y}\right)^2 + y^2 \right]^{1/2}} \; \hat{y} \; .$$

It is interesting to see what happens as the distance y from the line charge gets really large. In this case, the term $(l/y)^2$ in the denominator vanishes leaving

$$\vec{E} = \frac{Q}{4\pi\epsilon_o y^2} \; \hat{y} \; , \tag{2.17}$$

where we have let Q equal λl the total charge on the rod. This result is just what one would get a distance z from a point charge Q using Coulomb's law less the test charge. So we see that at large distances the finite line charge has an $\vec{E}$ field of a point charge Q.

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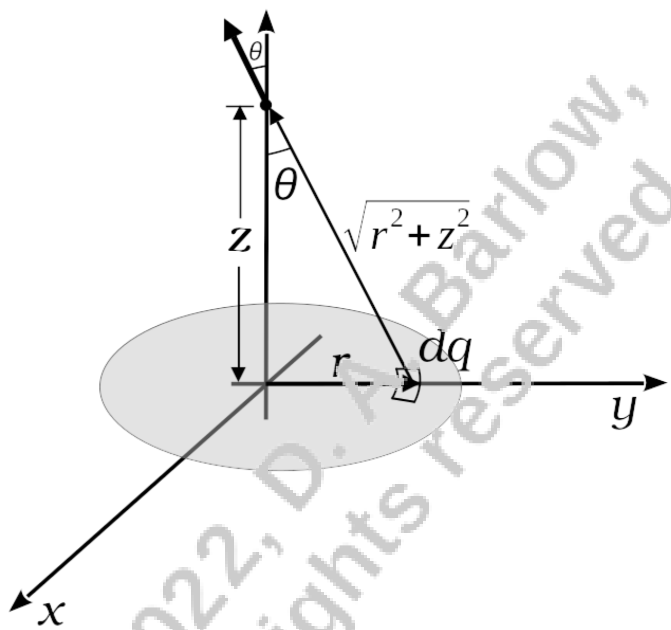
Example 2.2

Find the electric field vector a distance z above the center of a uniformly charged disk of radius R which lies in the xy plane and is centered at the origin. The disk has area charge density σ. The situation is depicted below.



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Here we use the differential for area in polar coordinates as discussed in Chapter 1. Therefore, dq will be:

$$dq = \sigma r dr d\phi \ .$$

Before going further, we note that the integral in Eq. (2.14) can be greatly simplified by taking advantage of symmetry. Notice how in the figure above as the integral sum of dq precedes around the disk the vector contributions in the plane of the disk will sum to zero. Therefore, we only require the vertical component of $d\vec{E}$ as we integrate over the disk. The vertical component is extracted from the integral when we multiply our integrand by $\cos\theta$. From the figure above we see that

$$\cos\theta = \frac{z}{\sqrt{r^2+z^2}} \ ,$$

and

$$r = \sqrt{r^2+z^2} \ .$$

Putting all of this into Eq. (2.14) gives

$$\vec{E} = \frac{1}{4\pi\epsilon_0}\int_0^R\int_0^{2\pi}\frac{\sigma r dr d\phi}{(r^2+z^2)}\frac{z}{\sqrt{r^2+z^2}}\ \hat{z} = \frac{1}{4\pi\epsilon_0}\int_0^R\int_0^{2\pi}\frac{\sigma z r dr d\phi}{(r^2+z^2)^{3/2}}\ \hat{z} \ .$$

The integral over ϕ yields 2π and the integral over r can be found in a table. This leads to

$$\vec{E} = \frac{z\sigma}{2\epsilon_0}\int_0^R\frac{r dr}{(r^2+z^2)^{3/2}}\ \hat{z} = \frac{z\sigma}{2\epsilon_0}\left[-\frac{1}{\sqrt{(r^2+z^2)}}\right]\ \hat{z}\Bigg|_0^R = \frac{\sigma}{2\epsilon_0}\left[1-\frac{z}{\sqrt{(R^2+z^2)}}\right]\hat{z} \ .$$

PROBLEMS

All charges are static in the following problems.

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2.4 A thin circular ring of radius R which holds uniform linear charge density λ lies in the xy plane centered at the origin. What is the electric field vector a distance z above the center of the loop?

2.5 For the charged disk discussed in Example 2.2, what does the final expression for $\vec{E}$ reduce to when $R >> z$?

2.6 Find the electric field vector a distance z above the center of an infinite straight line of uniform linear charge density that lies along the x axis. What does your final expression reduce to when $z \to \infty$?

2.7 A uniform surface charge density σ is on a square plate of side length l which lies in the xy plane. If the plate is centered at the origin what is the electric field vector a distance z above the center of the plate? What does your final expression reduce to when $z \to \infty$?

Even though the shapes of the charge distributions considered thus far have been favorable for the application of Eq. (2.14), one could use this formula to determined the electric field for any static charge distribution. These could be distributions with irregular shapes and cases where the charge density varies with position. However, you may have to resort to numerical techniques to evaluate the resulting integral. In the next section, we will explore a method whereby the electric field due to certain irregular static charge distributions can be easily obtained.

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2.3 Gauss's Law

In this section we will consider a powerful theorem of electrostatics called *Gauss's Law.* Gauss's law relates the flux of the electric vector field through a closed surface to the static charge within the surface. More specifically, the flux of $\vec{E}$ through a closed surface is directly proportional to the total charge enclosed Q_{enc}. Mathematically speaking we have

$$\oint_S \vec{E} \cdot d\vec{A} = \frac{Q_{enc}}{\epsilon_0} \ . \tag{2.18}$$

Here the constant of proportionality is the inverse of the permittivity of free space. We can learn more about Eq. (2.18) by assuming that the total charge enclosed can be given by the integral of the charge density, ρ, over the volume, τ, which the surface, S, encloses. That is,

$$Q_{enc} = \int_\tau \rho \ d\tau \ . \tag{2.19}$$

Using Eq. (2.19) in Eq. (2.18) leads to

$$\oint_S \vec{E} \cdot d\vec{A} = \int_\tau \frac{\rho}{\epsilon_o} \ d\tau \ . \tag{2.20}$$

Now, the fundamental theorem for divergences, from Chapter 1, can be used to convert the integral of a flux through a closed surface, on the left of Eq. (2.20), into an integral of a divergence of the electric field through the volume τ enclosed by the surface S. That is

$$\int_\tau \vec{\nabla} \cdot \vec{E} \ d\tau = \int_\tau \frac{\rho}{\epsilon_o} \ d\tau \ . \tag{2.21}$$

The above equality implies that the integrands must be equivalent so that

$$\vec{\nabla} \cdot \vec{E} = \frac{\rho}{\epsilon_o} \ . \tag{2.22}$$

This result is referred to as the differential form for Gauss's law while Eq. (2.18) is called the integral form.

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Example 2.3

An electric field in spherical coordinates is given by

$$\vec{E} = \frac{K}{r^n}\ \hat{r}\ ,$$

where n is a positive integer and K is a constant with SI units appropriate for the given value of n. Use Gauss's law to find the corresponding charge density for this field.

We use Eq. (2.22) in spherical coordinates. Taking a peek at a divergence formula in an Appendix, we see that with $\vec{E}$ only a function of r, the divergence only has one term, which is

$$\vec{\nabla}\cdot\vec{E} = \frac{1}{r^2}\frac{\partial}{\partial r}r^2\left(\frac{K}{r^n}\right)\ .$$

Taking the required derivative yields

$$\vec{\nabla}\cdot\vec{E} = \frac{(2-n)K}{r^{n+1}}\ .$$

Inserting this into Eq. (2.22) and solving for ρ gives our result

$$\rho = \frac{(2-n)K\epsilon_o}{r^{n+1}}\ .$$

Note how $\rho = 0$ when $n = 2$.

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PROBLEMS

2.8 An electric field in spherical coordinates is given by

$$\vec{E} = E_o e^{-\alpha r}\ \hat{r}\ ,$$

where E_o and α are constants. Show that the corresponding charge density is given by

$$\rho = \epsilon_o E_o\left(\frac{2}{r} - \alpha\right)e^{-\alpha r}\ .$$

2.9 In Example 2.3 we found that $\rho = 0$ for $n = 2$. Oddly, $\vec{\nabla} \cdot \frac{K}{r^n}\,\hat{r} = 0$ for $n = 2$ as well. However, by the fundamental theorem for divergences it must be that

$$\int_{\tau} \vec{\nabla} \cdot \vec{E}\, d\tau = \oint_{S} \vec{E} \cdot d\vec{A}\ .$$

Show that the integral $\oint_{S} \frac{K}{r^2}\,\hat{r} \cdot d\vec{A}$ for a spherical surface of radius R centered about the origin is in fact **not zero**, but rather equals $4\pi K$. Speculate as to how $\rho = 0$ yet there is some non-zero divergence for this field. That is, what could be generating the field?

2.10 Using Coulomb's law, and the definition for the electric field, $\vec{E} = \vec{F}/q$, write the expression for the electric field due to a point charge Q placed at the origin in spherical coordinates.

In the preceding example problem, Gauss's law, in the form of Eq. (2.22), was used to find a charge density given a known electric field. Gauss's law, in the form of Eq. (2.18), is more commonly used to determine the electric field due to a known static charge distribution. The central idea in this approach is to surround some static charge distribution with a fictitious closed surface S. We call this surface a *Gaussian surface.* Since the electric vector field is involved in an integral for flux in Eq. (2.18), this method is most useful when there is a high degree of symmetry in the problem and the direction of the electric field vectors are known at the Gaussian surface. The shape of the Gaussian surface is arbitrary, it can be of any style one chooses so long as it fully encloses the charge of interest. But, as you will see, certain Gaussian surfaces are ideal for particular cases and as you gain ex-

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perience you will become more adept at making the best choice for a given static distribution.

To demonstrate the application of Gauss's law, we will consider a single fixed point charge q in free space. We surround the charge with a spherical shell of radius r with the point charge at its center. You may have seen this result already in practice Problem 2.10 where Eq. (2.11) and Coulomb's law were used to derive it. Everywhere on the surface of the sphere the electric field vectors will be of the same magnitude since they vary only with r. Further, since the electric field vectors are all radially directed they will all be in the same direction as $d\hat{A}$, that is, the $\hat{r}$ direction. Therefore, in this case the integrand of the integral in Eq. (2.18) becomes EdA and we get

$$\oint EdA = \frac{q}{\epsilon_o}$$

Since E is constant over this surface it comes out of the integral and $\int dA$ simply leads to the surface area for a sphere of radius r. Eq. (2.18) now becomes

$$E(4\pi r^2) = \frac{q}{\epsilon_o} \ . \tag{2.23}$$

Solving this for E then leads to the expression for the electric field vectors a distance r from a point charge q:

$$\vec{E} = \frac{q}{4\pi\epsilon_o r^2} \ \hat{r} \ . \tag{2.24}$$

In the next example we will use Gauss's law to find a useful expression for the magnitude of the electric field between two equal and oppositely charged plates.

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Example 2.4

Use Gauss's law to find an expression for the electric field strength between two parallel, infinitely large, flat, equal and opposite uniformally charged conducting plates. Let the charge on the plates be given by a constant area charge density σ.

Even though we are considering infinitely large plates, this approximation turns out to be quit reasonable for regions within the interior of the parallel plate system. Such a region is shown in the figure below.

Here, for an enclosing surface we will use what is called a *Gaussian pillbox*. Consider the left positively charged plate of a parallel plate capacitor. The plate holds area charge density σ. The pillbox is a cylinder of circular cross-section. Let each circular end of the pillbox have area A. The cylinder is then placed so that the charged plane bisects it and a selected amount of the positive charge on the plate is then enclosed within it. This setup is depicted in the figure below.

The amount of charge enclosed by the pillbox is simply σA. Some of the electric field vectors between the charge plates are shown in the figure. Notice how they are all perpendicular to the right end surface of the pillbox. This enclosing surface has three surfaces to consider. At the left end we assume that the there is no electric field outside of the parallel plate system so that the flux through this surface is zero. The second surface to consider is the side of the cylindrical pillbox. Here all of the electric field vectors are perpendicular to any unit vector that would be normal to this surface so that the inner product in Gauss's law yields zero on this entire surface. Therefore, there is no flux through this surface. Finally, the right end of the cylinder is the only surface with a non-zero flux. Here, the electric field vectors are in the same direction as a unit vector normal to this surface so that Gauss's law in this case becomes

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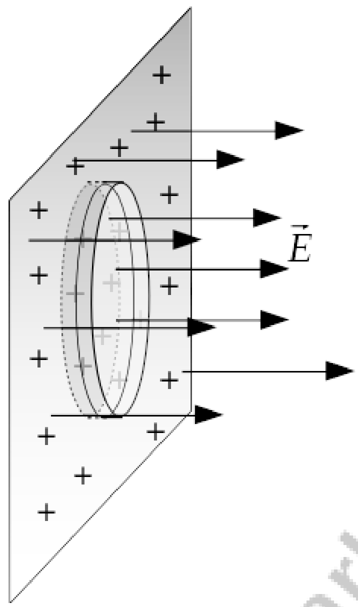

Figure 2.5: A Gaussian pillbox encloses charge on the left plate of a charged capacitor. (Right side negatively charged plate not shown.)

$$\int_S E\, dA = \frac{\sigma A}{\epsilon_o} \ .$$

E is uniform and constant so it comes outside of the integral and $\int dA$ simply yields A so that

$$EA = \frac{\sigma A}{\epsilon_o} \ .$$

Solving for E we get

$$E = \frac{\sigma}{\epsilon_o}$$

Interestingly, the electric field is constant everywhere between the plates. That is, it does not vary with position.

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In concluding this section, it should be mentioned that the location of excess charge in solid materials is important for the proper application of Gauss's law. For solid conducting materials the excess charge always resides on the surface. Why is this the case? Imagine some excess positive charge placed on a solid metal sphere. Since charge is free to move around where will it end up? This then leads to another question, if the excess charge resides on the surface of a conductor, what is the electric field within its interior? On the other hand, if you encounter a situation where excess charge is uniformly distributed within a solid it must be that the object is an insulator.

Another item of note is that if an outside charge is placed within the vicinity of a conductor without making physical contact, equal and opposite charge within the conductor will flow to the surface nearest to the outside charge. We call this surface charge an *induced* charge.

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PROBLEMS

2.11 A solid sphere of radius R has uniform volume charge density ρ. Use Gauss's law to find expressions for the electric field inside and outside of the sphere. Sketch a plot of E for all regions. Hint: when your Gaussian surface is within the sphere of uniform charge, it encloses some fraction of the total charge.

2.12 Use Gauss's law to find an expression for the electric field of an infinite, straight line of charge of uniform linear charge density λ. Hint: Use a Gaussian cylinder.

2.13 Use Gauss's law to find the electric field due to a single infinite flat plate of uniform charge density σ. Hint: the single charged plate will have non-zero field on both sides.

2.14 A solid conducting sphere of radius b has a hollow spherical cavity of radius a perfectly centered within the conducting sphere. Obviously, $a < b$. A point charge $+Q$ is fixed in place in the very center of the cavity. Use Gauss's law to find the electric field in all regions.

2.4 The Electrostatic Field

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In the previous sections you were introduced to the definition of the electric field and how to arrive at mathematical expressions for this vector field for various static charge distributions. In this section we want to delve deeper into the nature of these results and in doing so we will discover certain regular characteristics for these vector fields, fields which are referred to as *electrostatic* fields.

To reveal one such property let us consider the electric field of a point charge q from the previous section.

$$\vec{E} = \frac{kq}{r^2}\,\hat{r}\ , \tag{2.25}$$

where k is the Coulomb constant. It is of interest to consider the line integral of this $\vec{E}$ between to points r_1 and r_2 where $r_1 < r_2$. This is just

$$\int_{r_1}^{r_2} \vec{E}\cdot d\vec{l}\ . \tag{2.26}$$

In the case of Eq. (2.25), $d\vec{l} = dr\hat{r}$. So that upon using

Eq. (2.25) in Eq. (2.26) we get

$$\int_{r_1}^{r_2} \frac{kq}{r^2}\ \hat{r} \cdot dr\hat{r} = \int_{r_1}^{r_2} \frac{kq}{r^2} dr \ . \tag{2.27}$$

This evaluates to

$$\int_{r_1}^{r_2} \frac{kq}{r^2} dr = -kq \left(\frac{1}{r_2} - \frac{1}{r_1} \right) \ . \tag{2.28}$$

Now, suppose rather than consider the above line integral from r_1 to r_2, we compute it around a closed circular loop of radius R where $r_1 = r_2 = R$. From eq. (2.28) we see that this evaluates to zero. That is,

$$\oint \frac{kq}{r^2} dr = 0 \ . \tag{2.29}$$

From the fundamental theorem for curls this implies that

$$\int_S (\vec{\nabla} \times \vec{E}) \cdot d\vec{A} = 0 \ , \tag{2.30}$$

and therefore it must be that for a fixed point charge, $\vec{\nabla} \times \vec{E} = 0$.

This is an interesting fact but it would be more useful if we could verify it for any number of fixed point charges. Recall from Eq. (2.7) that the principle of superposition holds for the Coulomb force due to multiple fixed point charges. On dividing Eq. (2.7) by a positive test charge placed at some point P near N fixed point charges we get

$$\vec{E} = \vec{E}_1 + \vec{E}_2 + \vec{E}_3 + \cdots + \vec{E}_N \ . \tag{2.31}$$

Taking the curl of both sides of Eq. (2.31) gives

$$\vec{\nabla} \times \vec{E} = \vec{\nabla} \times \vec{E}_1 + \vec{\nabla} \times \vec{E}_2 + \vec{\nabla} \times \vec{E}_3 + \cdots + \vec{\nabla} \times \vec{E}_N \ . \tag{2.32}$$

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Since all fields above are due to point charges, and we found that the curl of the electric field due to one point charge is zero, it must be that the right side of Eq. (2.32) equals zero. We than can state that in general, **The curl of an electrostatic field is zero**.

PROBLEMS

2.15 Which of the following fields can be electrostatic fields?

a) $\vec{E} = 2\ \hat{x} + 3y^2\ \hat{y}$ N/C
b) $\vec{E} = \sin x\ \hat{x} + 3y^2\ \hat{y}$ N/C
c) $\vec{E} = \frac{K}{r^3}\ \hat{r}$ N/C
d) $\vec{E} = 2xy^3\ \hat{x} + (y-2)x^2\ \hat{y} - 2xz^4\ \hat{z}$ N/C

2.16 Consider the following electric fields in SI units:

$$\vec{E} = \frac{K}{r^n}\ \hat{r}\ ,$$

where K is a constant and n is a positive integer. For which values of n will $\vec{E}$ be an electrostatic field?

2.17 For what values of a and b can the following be an electrostatic field?

$$\vec{E} = 2ax^2\ \hat{x} + bxy^4\ \hat{y}\ .$$

2.18. Let $\vec{E} = 3x\ \hat{x} - 4\ \hat{y}$ N/C. Compute the line integral

$$\int \vec{E} \cdot d\vec{l}$$

along the path of the horizontal line from $x = 0$ to $x = 4$ with $y = 0$ and then along the vertical line from $y = 0$ to $y = 3$ while $x = 4$.

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2.19 A charge of +2.0 μC is fixed at the origin of a three dimensional Cartesian system. What is the net Coulomb force acting on this charge due to charges +3.0 μC fixed at $(0, 1, 2)$, -2.0 μC fixed at (1, 3, -2) and +4.0 μC fixed at (1, 1, 1).

2.20. Use Gauss's law to find the electric field inside and outside of a hollow sphere of radius R which carries a uniform surface charge density σ. Sketch a plot of the magnitude of the electric field as a function of the radial distance from the center of the sphere.

2.21. Use Gauss's law to find the electric field inside and outside of a solid sphere of radius R which carries a uniform volume charge density $\rho = \rho_o(1 - e^{-\alpha r})$, where ρ_o and α are constants. Sketch a plot of the magnitude of the electric field as a function of the radial distance from the center of the sphere.

2.22. A solid sphere of radius R has a charge density $\rho = kr$ where k is a constant and r the radial coordinate. Compute the total charge in the sphere.

2.23 For the sphere in the previous problem, use Gauss's law to find the electric field inside and outside the sphere.

2.24 Three infinite planes with surface charge density σ are configured parallel to the xz plane in a three dimensional system. One is at $y = 0$, another at $y = a$ and the third at $y = 2a$. Use Gauss's law, and the principle of superposition, to find the electric field in the regions $y < 0$, $0 < y < a$, $a < y < 2a$ and $y > 2a$.

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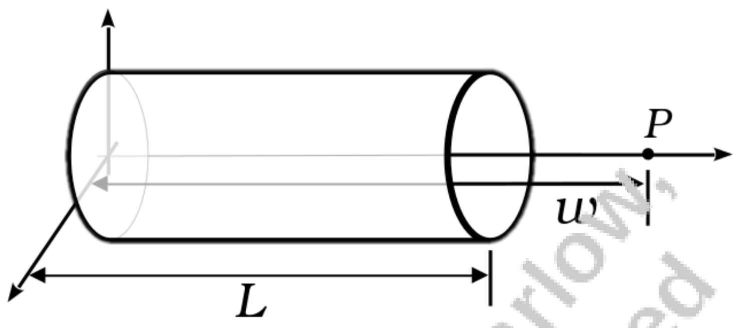
2.25 Find the electric field at a distance w from the origin along the axis of a uniformly charged thin hollow cylinder of radius R and length L. Let the total charge on the cylinder be Q.



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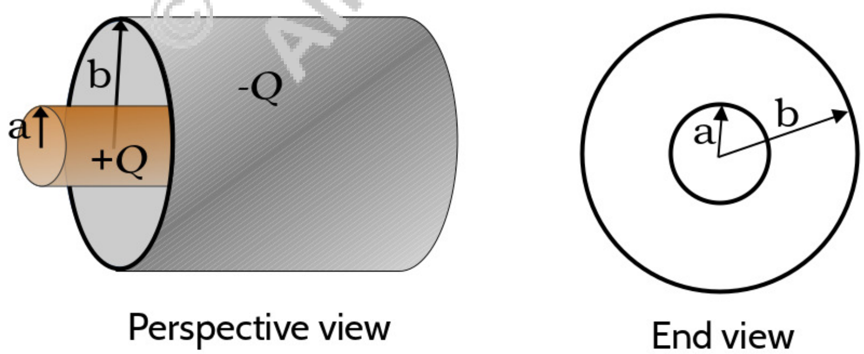
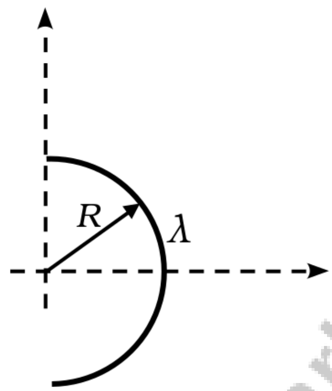
2.26 A *coaxial* cable, of arbitrary length, has a solid cylindrical conductor that holds excess charge $+Q$ and has radius a. This is surrounded by a larger thin conducting cylinder of radius b which acts as a shield for the inner conductor and holds excess charge $-Q$. Use Gauss's law to find expressions for the electric field in the regions $r < a$, $a \leq r \leq b$ and $r > b$.



2.27 Is $\vec{E} = 2x^2\ \hat{x} + (y+2)^3\ \hat{y}$ N/C an electrostatic field? If so, find the corresponding charge density ρ.

2.28 A thin semicircular wire holds uniform linear charge density λ. This section of wire is shown in a coordinate system below. What is the electric field at the origin?



2.29 An electrostatic field is $\vec{E} = 3\,\hat{x} + 2y\,\hat{y}$. Compute the line integral

$$\oint \vec{E} \cdot d\vec{l}$$

around the closed path of the horizontal line from $x = 0$ to $x = 4$ while $y = 0$ and then along the vertical line from $y = 0$ to $y = 3$ while $x = 4$. Then along the horizontal line from $x = 4$ to $x = 0$ while $y = 3$ then back to the starting point along the vertical line $y = 3$ to $y = 0$ for $x = 0$.

2.30. A solid sphere of radius R has a charge density $\rho = k(R - r)$ where k is a constant and r the radial coordinate. Compute the total charge in the sphere. Use Gauss's law to find the electric field inside and outside the sphere.

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